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Strength of Columns Elastically Restrained and Eccentrically Loaded

by

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Summary

The strength of simply supported columns, concentrically and eccentrically loaded, can be calculated by recognized methods. In the case of eccentrically loaded columns these methods of necessity account for the influence of plastic deformation. Most actual columns do not represent isolated members, but are more or less rigidly connected at the ends to other structural components. To approach this condition, the present paper deals with theory and tests relating to eccentrically loaded columns with equal elastic restraints at both ends.

A precise analytical method and a simplified method are given for calculating the buckling loads for such columns for the case of equal end eccentricities.

In the precise method the length between inflection points of such columns was obtained by assuming the buckled shape to be a sine wave with respect to the pressure line of the compressive forces. In this manner an equivalent, simply supported column is obtained. To determine its ultimate load, the actual rather than the approximate deflected shape must be taken into account. This shape depends on the stress-strain diagram of the material.

The simplified method reduces the problem to that of analyzing an equivalent simply supported eccentrically loaded column. Use is made of an "effective eccentricity" in a manner analogous to that developed at Cornell University by Miss Annabel Lee for investigating the elastic stresses in such columns. General formulas are given which greatly reduce the amount of computation necessary for determining these buckling loads. Torsional buckling is not considered.

These ultimate buckling loads may be up to 100% higher than those which cause first yielding in the outer fiber, provided local failure by flange buckling does not occur. Column provisions in current codes are generally based on those loads which cause incipient yielding, rather than on the ultimate loads.

Tests are reported on 18 columns, half of them on square bars and the other on I-sections. To investigate the influence of residual stresses, half of the specimens were annealed. Columns were tested with fixed amounts of elastic restraint and with various eccentricities, up to $e/r = 2$. Ultimate loads were determined as well as loads which caused first local yielding. Stress-strain tests in tension and compression were made on the column material.

Experimental ultimate loads and loads which caused first yielding are compared with the values predicted by the theory. Very satisfactory agreement is obtained throughout.

INTRODUCTION

The performance of isolated columns, both concentrically and eccentrically loaded, is well understood and their carrying capacities can be computed if the stress-strain relation of their material is known. Most compression members, however, are connected at their ends to other parts of the structure. If they are loaded concentrically, methods are available to calculate their buckling loads (see e.g. Ref. 8). In reality most columns are subject to eccentric compression, be it from imperfection eccentricities, as in the case of many trusses, or from end moments transmitted by abutting members. In his recent book Buckling Strength of Metal Structures (McGraw-Hill, 1952) F. Bleich states: "An elaborate study of eccentrically loaded columns with end

restraints must be considered as the first step in an investigation of the behavior of columns as part of a rigid frame or in estimating the effect of secondary stresses in trusses upon the buckling strength of its compression members." As part of such a study this paper reports an analytical and experimental investigation of columns with equal elastic restraints and with equal eccentricities at both ends.

An exact analysis of this type, including plastic deformation of the column, is exceedingly cumbersome. The only exact work on this problem known to the authors is a paper by Chwalla[1], which is not capable of generalization for practical use. It is a grapho- analytical method which is very laborious and was worked out only for eccentric buckling stresses of 21300 psi (1500 kg/cm^2).

In the present paper a general analytical method for analyzing this type of column is indicated first. This method serves as a check for the development of a simplified, approximate method, which is based on the concept of the equivalent eccentrically loaded simply supported column, as used in Miss Lee's Cornell University thesis[2] for calculating the load at which yielding starts in the outer fibers in eccentric, end restrained columns. Similar methods were developed for eccentrically loaded columns with any degree of end restraint, equal or unequal, and with any amount of end

eccentricity, equal or unequal. They are not reported in this paper but can be found in Ref. 3.

ANALYTICAL METHOD

The reason why an exact determination of the eccentric buckling stress of simply supported as well as of elastically restrained columns is so laborious is that eccentric buckling always occurs in the plastic region. Let Fig. 1a represent the stress-strain curve, in tension and compression, of the eccentrically loaded column of Fig. 1b. With increasing bending moments P.y the outer fiber strains in the central portion of the column, where moments are large, surpass the proportional limit strain e_p. Consequently, near and at failure the stress distribution over a cross-section near mid-length is the plastic distribution of Fig. 1c, while near the ends, where moments are small, the elastic distribution of Fig. 1d will usually prevail.

As long as only elastic deformations occur, the required load P to hold the column in equilibrium will steadily increase and would approach asymptotically the Euler load (Curve A in Fig. 2). Only when plastic deformations begin to occur, by which the effective flexural deformation modulus of the column is diminished, the load-deflection diagram of the column will deviate from Curve A (at point C in Fig. 2) and follow a lower curve B. Since in most cases the effec-

tive modulus of the material decreases gradually with increasing curvature, curve B will deviate more and more from curve A, and at a certain point M it will attain a maximum value $P_{cr} = A\sigma_{cr}$. This is the critical or buckling load P_{cr} of the column while σ_{cr} is the maximum average stress P_{cr}/A it can sustain, the eccentric buckling stress.

In the case of a simply supported eccentrically loaded column, for example, the condition that internal and external moments are equal requires that (Fig. 3)

$$-\eta EI\frac{d^2y}{dx^2} = Py \qquad (1)$$

where the X-axis is taken along the axis of thrust. Here η is the reduction factor for plasticity which is unity in the elastic domain. In the plastic domain and for infinitely small deflections, as with concentric buckling, this reduction coefficient can safely and sufficiently accurately be assumed as E_t/E, where E_t is the tangent modulus.[4] It is invariant with x and thus a constant in Eq. (1), which is consequently satisfied by a sine wave.

If, however, the deflections are finite, as in eccentric buckling and also for finite deflections of concentrically loaded columns subsequent to buckling, η is no longer independent of x. In Fig. 3, for example, it is smaller in the center of the column, where the curvature is higher, and larger at the ends. Hence the curvature of the column at the

center will be larger than that of a sine wave of equal maximum deflection while near the ends it will be smaller.

The reason why an exact determination of eccentric buckling stresses of simply supported as well as of elastically restrained columns is so laborious is precisely the above fact, namely that in the plastic domain the deflection curve is no longer the branch of a sine wave.

These facts were taken into account in Chwalla's calculations of eccentrically loaded and simply supported columns as well as of the equilibrium load of concentrically loaded columns with finite post-buckling deflections.[5) 6)] For eccentrically loaded and elastically restrained columns, however, this method does not lead to directly applicable results.

Fig. 4 shows two cases of elastically restrained columns considered in the present paper. For the case of Fig. 4a the form of the deflected column is sketched in Fig. 5a. Points G and H are inflection points. Since the moments are zero at these points, the line of thrust of force P passes through G and H. The distance between the inflection points G and H, which henceforth will be called the equivalent length, will decrease with increasing external force P while the amplitude y_m increases. This follows from the fact that the compressive resistance is equal to that of the hinged column

GH with length ℓ and deflection y_m. For a hinged column with constant length the resistance in the elastic range remains practically constant with increasing deflection y_m. In the plastic range the resistance of a hinged column decreases with increasing y_m, because in this case the effective plastic deformation modulus decreases. Hence the only way for the elastically restrained column to resist an increasing load P is to diminish its equivalent length ℓ, so that its equilibrium load can still increase because its equivalent length decreases. (It is simple to demonstrate this behavior in the elastic range from Eq. (7) and (m) of Ref. 8. In the plastic range the progressive decrease in equivalent length is further accelerated by the plastification of the column.) At last, however, the decrease of the equivalent length ℓ can no longer cancel the decrease of the effective modulus η E and the equilibrium load will begin to decrease, after having reached an extreme value, the eccentric buckling load P_{cr} (compare Fig. 2).

If for any equivalent length ℓ = GH the pertinent maximum deflection y_m could be determined, the resulting equilibrium load and hence P_{cr} could be calculated from the following reasoning. Since the eccentrically loaded column was originally straight, the bending moments in part GH will be about the same as in an originally straight and concentrically

loaded column after buckling with the same maximum deflection y_m. A graph of resistance vs. deflection of originally straight simply supported rectangular columns after buckling has been calculated by Chwalla,[6] using von Karman's method of double integration,[7] for a steel with a yield stress of 38,000 psi (2700 kg/cm^2) and an elastic limit of 27,000 psi (1900 kg/cm^2). The slenderness-buckling stress relation for concentric buckling of columns of the same steel is given in Fig. 6, where σ_{cE} is the buckling stress according to Engesser and Shanley and σ_{cK} is that according to von Karman.

The first mentioned graph had to be corrected for ratios y_m/h smaller than 0.1*, since it was based on the concept of unloading of originally plastically compressed fibers (i.e. the von Karman theory). In the case of eccentric compression, where the compressive force and the bending moments increase simultaneously, such unloading is non-existent or negligible, so that in any section the stress distribution is determined by the original stress strain diagram of the material (Fig. 1c). Accordingly this leads to the Engesser-Shanley load[4]

$$\sigma_{cE} = \frac{\pi^2 E_t}{(L/r)^2} \qquad (2)$$

for zero eccentricity. For ratios y_m/h larger than 0.1 practically no unloading of initially plastically deformed fibers occurs also in the case of concentrically loaded columns, so that no correction was necessary for this range. The result-

* h = thickness of column in direction of buckling

ing graph, Fig. 7, gives the mean equilibrium stress σ_a for a given equivalent slenderness l/r (Fig. 5a) and a given maximum deflection y_m for a steel with a yield stress of 38000 psi as used in Chwalla's work.

Although the actual deflected shape has to be taken into account for determining the equilibrium load for a given finite deflection y_m of a column with a half wave length l (Fig. 5a) (see Ref. 3), for determining the half wave length l itself it is sufficiently accurate to consider the deflection curves as branches of sine waves. This leads to the following relation between the three unknowns, that is, the half wave length l, the mean stress σ_a and the maximum amplitude y_m of the half wave l:

$$y_m = \frac{e}{\cos\frac{\pi}{2}\frac{L}{l} + \frac{3\pi E}{(4r)(4r)\sigma_a}\sin\frac{\pi}{2}\frac{4}{l}} \qquad (3)$$

Since Fig. 7 gives another relation between these three unknowns, by choosing several values l the corresponding y_m and σ_a may be found from Eq. (3) and Fig. 7 by trial and error and plotted such as to obtain a graph like curve CB in Fig. 2, where $P = A\,\sigma_a$. The maximum value of P is the critical load of the eccentrically loaded column. For example, for a case as sketched in Fig. 4a exact data are available in Ref. 1 for $L/r = 119.6$, $L_1/r = 50$ and $e/r = 2$ or $e/h = .578$, leading to the exact result $\sigma_a = 21{,}300$ psi (1500 kg/cm^2). From the slightly approximate Eq. (3) and Fig. 7

one obtains the values listed in Table 1, which represent associated pairs of values of y_m and σ_a near point M of a curve such as CMB on Fig. 2.

Table 1

l/r	y_m/h	σ_a (psi)
95	.253	21080
90	.265	21160
85	.286	21160
80	.310	20860

Hence the maximum stress reached is the eccentric buckling stress and is found to be σ_{cr} = 21160 psi, in good agreement with the exact value of 21300 psi. Comparisons such as this document the adequacy of the method based on Eq. (3) and Fig. 7.

This method was used to check the simplified method, which will be described below.

SIMPLIFIED METHOD

To arrive at a simpler approximate design procedure, Miss Lee's method[2] for determining the load at which outer fiber yielding starts in eccentrically compressed elastically restrained columns, was extended to eccentric buckling. According to Miss Lee's work the outer fiber stress in a column AB (Fig. 5a) will reach the yield stress at practically the same load as in a simply supported column with a length equal to the effective length cL of the concentrically loaded column AB and with an effective eccentricity ε (Fig. 5b).

To find ε , the external moments Pe (Fig. 4) are distributed to the members AB and to the elastic restraints, neglecting the compressive force in the restraining members.

As stated above in connection with Figs. 1 and 2, eccentric buckling always occurs in the plastic region. Hence it may be assumed that this method may also be used to determine the eccentric buckling stress, if, for determining the effective length cL and the effective eccentricity ε of the corresponding simply supported column a proper plastic reduction factor η is used, to account for the decrease of the flexural rigidity EI of the buckling member AB (Fig. 4).

The procedure of the Simplified Method, consequently, is as follows: By means of the proper reduction factor η determine the effective length for concentric buckling (cL of Fig. 5b) of the simply supported column equivalent to the restrained column of Fig. 5a. Find the concentric buckling stress $\sigma_{c\varepsilon}$ of this equivalent column from Eq. (2). Next, determine the equivalent eccentricity ε of the simple column, Fig. 5b, corresponding to the real eccentricity e of the restrained column, Fig. 5a, again using η . Then the eccentric buckling stress σ_a , which of course is smaller than the critical stress for concentric buckling, $\sigma_{c\varepsilon}$, can be found for the equivalent, simple column,

$$\sigma_{cr} = \sigma_{c\varepsilon}/\gamma$$

where γ is a division factor larger than one. This

division factor depends on the effective eccentricity ratio $\mathcal{E}/k_c$ (k_c = core radius r^2/c) and the effective slenderness cL/r. It is a quantity similar to β used by F. Bleich on p. 41 of Ref. 9.

The determination of the various quantities which enter this method was possible only by fitting expressions to the results of the few and limited exact determinations available in the literature, to the Analytical Method discussed above, and to the test results reported hereafter, and is given below.

(a) Determination of η for the steel used in Ref. 1.

The reduction factor η was determined by comparison (1) with the exact calculations given in Ref. 1 for some cases, (2) with the semi-exact analytical method discussed above and (3) with the test results described hereafter. The exact calculations of Ref. 1 refer to the case of Fig. 4a and to a steel with a yield stress of 38000 psi (2700 kg/cm^2) for which Fig. 6 gives the relation between the slenderness ratio L/r and the concentric buckling stresses σ_{cK} and σ_{cE} according to von Karman and Engesser-Shanley, respectively.

σ_{cE} is given by Eq. (2) and

$$\sigma_{cK} = \frac{\pi^2 E_r}{(L/r)^2} \qquad (4)$$

where E_r is the reduced modulus. In Ref. 1 Eq. (4) was

used, but, as Shanley[4] showed later, Eq. (2) should be used.

It may be expected that the reduction factor η will vary with the buckling stress as well as with the slenderness. It will differ from the reduction factors E_t/E or E_r/E, which, from Eqs. (2) and (4), are

$$\frac{E_t}{E} = \frac{\sigma_{cE}(L/r)^2}{\pi^2 E} \quad \text{and} \quad \frac{E_r}{E} = \frac{\sigma_{cK}(L/r)^2}{\pi^2 E} \qquad (5)$$

E_t and E_r differ considerably for a given value of the buckling stress, since from Fig. 6, for a given buckling stress as ordinate the abscissa L/r, which appears in Eq. (5) to the second power, differs considerably for the two curves. However, for a given abscissa L/r, from Fig. 6, the ordinates σ_{cE} and σ_{cK} of the two curves, which appear in Eq. (5) to the first power only, differ very little, so that also E_t and E_r from Eqs. (5) will differ very little. This suggests to express η as a function of the effective slenderness cL/r rather than of the eccentric buckling stress. This is fortunate because the available exact data[1] are all for the same eccentric buckling stress of 21,300 psi (1500 kg/cm^2). The case investigated in Ref. 1 is that of Fig. 4a where members CA, AB and DB have the same rectangular cross sections. The numerical values of the parameters e/r, L_1/r and L/r used in Chwalla's ten cases are assembled

in Table 2, which also gives the v. Karman buckling stress σ_{cK} for concentric loading. The significance of the columns headed ε/k_c, γ' and σ_{cr} is explained below.

Table 2

Case	e/r	L_1/r	L/r	cL/r	ε/k_c	σ_{cK} (kg/cm²)	γ'	σ_{cr} (kg/cm²)
1	0.25	200	123.8	99	0.224	2040	1.41	1445
2	0.25	100	156.5	105	0.13	1950	1.31	1490
3	0.25	50	188.9	110	0.065	1795	1.20	1495
4	0.25	19.1	217.0	115	0.024	1658	1.08	1530
5	0.25	0	241.2	121	0	1500	1	1500
6	2	200	48.1	35	1.35	2645	1.79	1480
7	2	100	77.2	51	0.97	2570	1.74	1480
8	2	50	119.6	71	0.59	2410	1.63	1480
9	2	19.1	176.2	94.5	0.23	2110	1.41	1500
10	2	0	241.2	121	0	1500	1	1500

From these data the reduction factor η to be used in the proposed Simplified Method can be determined so as to lead to the proper eccentric buckling stress of 21,300 psi (1500 kg/cm^2). In more detail this method for finding η was as follows:

The problem is to reduce the elastically restrained column with eccentricity e (Fig. 4a) to a simply supported column of length cL and with eccentricity $\mathcal{E}$ (Fig. 5b). For the latter case the plastic reduction factor $\gamma' = \frac{\sigma_{cK}}{\sigma_{cr}}$ is known from Ref. 6. By adjusting η so that for a simply supported column of length cL and effective eccentricity $\mathcal{E}$ the correct buckling stress $\sigma_{cr} = \sigma_{cK}/\gamma'$ is obtained

(1500 kg/cm^2 in the case of Table 2), the problem is solved.

First the actual spring constant β of the elastic restraints given by the adjacent members CA and BD is calculated, being

$$\beta = \frac{3EI}{L_1} \tag{6}$$

Then the effective length cL for concentric buckling of column AB can be found from Graph II of Ref. 8, where β/EK now has to be replaced by $\beta/\eta EK$, and K = I/L. Hence from Eq. (6)

$$\beta/\eta EK = \frac{3\,L/L_1}{\eta} \tag{7}$$

where η is still an unknown function of cL/r. If at first the ratio cL/r is estimated, η can be determined. Then, with $\beta/\eta EK$ from Eq. (7), from Graph II of Ref. 8 the factor c and hence cL/r is found. This must be equal to the value originally estimated. Thus c is found by trial and error and cL/r has been entered in Table 2. Then the equivalent eccentricity ε is calculated by distributing the external moments Pe, neglecting the influence of the axial forces in all members. Hence from Fig. 4a

$$\varepsilon = \frac{2\eta EK}{2\eta EK+\beta} \tag{8}$$

With the values cL/r and ε/k_c so obtained, where k_c is the core radius, the eccentric buckling stress of the column is

found as $\sigma_{cr} = \sigma_{cK}/\gamma'$, where σ_{cK} can be determined from Fig. 6 for the known ratio cL/r. The reduction factor γ' for eccentrically loaded and simply supported columns is found from the table on page 59 of Ref. 6 and has been entered in Table 2. It appeared that a good approximation of the actual buckling stress of 1500 kg/cm^2 is obtained if it is assumed that η' denoting a provisional value, is given by the following relations:

$$\left.\begin{array}{ll} \text{for } cL/r > 90 & : \ \eta' = 1 \\ \text{for } 90 > cL/r > 30 & : \ \eta' = 0.014\,\frac{cL}{r} - 0.26 \\ \text{for } 30 > cL/r > 20 & : \ \eta' = 0.16\left(\frac{cL}{30r}\right)^2 \end{array}\right\} \quad (9)$$

These relations have been plotted in Fig. 8 and have been used to calculate ε/k_c in Table 2, by means of Eq.(8). The values of γ' correspond to the respective values of ε/k_c and the last column, which gives the eccentric buckling stress $\sigma_{cr} = \sigma_{cK}/\gamma'$ is seen to be in excellent agreement with Chwalla's exact values 1500 kg/cm^2.

The above determinations refer only to one specific value of the buckling stress, as used by Chwalla, i.e. 1500 kg/cm^2 (21,300 psi.). To get some idea of the variation of η with the buckling stress, Figs. 2 and 3 of Ref. 6 have been used. These give plots of computed loads vs. deflections y_{om} of the type of Fig. 10 for simply supported, eccentrically loaded columns (Fig. 9) for a range of buckling

stresses from 9450 to 40,000 psi. (665 to 2820 kg/cm^2). They cover eccentricity ratios e/k_c = 1/8 and 2 (i.e. e/h = .0208 and .333) and several slenderness ratios L/r. From the values σ_{cr} and y_{om} at the point M (Fig. 10) of maximum load the equivalent bending modulus $\bar{\eta}E$ may be calculated as follows:

Assuming $\bar{\eta}$ as independent of x (Fig. 9) the equality between external and internal moments yields

$$P_{cr}(e+y) = -\bar{\eta}EI\,d^2(e+y)/dx^2 \qquad (10)$$

so that

$$(e+y) = (e+y_{om})\cos\frac{\pi}{l}x \qquad (11)$$

Insertion in both sides of Eq. (10) gives

$$\bar{\eta} = \frac{P_{cr}\,l^2}{\pi^2 EI} \qquad (12)$$

The boundary conditions at $x = \pm L/2$ require

$$(e+y_{om})\cos\frac{\pi}{l}\frac{l}{2} = e$$

or

$$\frac{\pi}{l}\frac{l}{2} = \cos^{-1}\frac{e}{e+y_{om}} \qquad (13)$$

From Eq. (12) and (13), using $I = Ar^2$

$$\bar{\eta} = \frac{P_{cr}}{EI}\,\frac{L^2}{4\left(\cos^{-1}\frac{e}{e+y_{om}}\right)^2} = \frac{\sigma_{cr}}{E}\,\frac{L^2}{r^2}\,\frac{1}{4\left(\cos^{-1}\frac{e}{e+y_{om}}\right)^2} \qquad (14)$$

Values thus obtained are plotted in Fig. 8 for the ratios e/k_c = 1/8 and 2 as functions of the effective

slenderness cL/r, which in the present case (Fig. 9) is equal to L/r.

Thus from η' and $\bar{\eta}$ in Fig. 8 and from comparison with the results of the above described semi-exact method and with the test results, it was concluded that the straight line through the origin and a point cL/r = 90, η = 1, as shown in Fig. 8, gives conservative values for the entire range and is a good approximation for η . In this connection it is pointed out that this straight line leads to safe buckling stresses because a higher η leads to lower critical stresses. Hence for a steel with a yield stress of 38000 psi

$$\left.\begin{array}{lll} \text{for } cL/r > 90 & : & \eta = 1 \\ \text{for } 90 > cL/r > 20 & : & \eta = (cL/r)/90 \end{array}\right\} \quad (15)$$

The determinations, above, refer only to the specific steel (σ_{ys} = 38,000 psi.) used in Ref. 1. To develop a method which is applicable to mild steel of any yield point it is necessary to generalize this information.

(b) General approximate formulas for concentric buckling stress σ_{cE}, division factor γ for eccentricity and plastic reduction factor η for steels with an arbitrary yield stress.

In order to be able to use the above Simplified Method for steels with an arbitrary yield stress, and taking account of the Engesser-Shanley rather than the von Karman stress

for concentric buckling, more general formulae are developed below.

In order to determine the concentric buckling stress σ_{cE} from Eq. (2) it would be necessary to use the proper tangent modulus E_t which varies with stress. Rather than to attempt a statistical evaluation of steel stress-strain diagrams, the inverse method will be used of finding an expression for σ_{cE} from known column curves for concentric buckling. (This method is similar to that used on pp. 53 - 55 of Ref. 9).

Assuming the proportional limit to be 10,000 psi below the yield stress σ_{ys}, the Engesser-Shanley concentric buckling stress for simply supported columns is approximated very well by the curve (Fig. 11).

$$\sigma_{cE} = \sigma_{ys} - 5000 \left\{ \frac{L/r}{(L/r)_E} + \left[\frac{L/r}{(L/r)_E} \right]^2 \right\} \qquad (16)$$

as shown by comparison between Fig. 6 and 11. Here $(L/r)_E$ is the slenderness for which the buckling stress is equal to the proportional limit, so that

$$(L/r)_E = \left(\frac{3 \cdot 10^6}{\sigma_{ys} - 10,000} \right)^{1/2} \qquad (17)$$

To calculate the eccentric buckling stress σ_{cr} from

$$\sigma_{cr} = \sigma_{cE}/\gamma \qquad (18)$$

it is necessary to determine the division factor γ for eccentricity.

The division factor $\gamma' = \sigma_{cK}/\sigma_{cr}$ related to σ_{cK} for simply supported columns, is given in the table on page 59 of Ref. 6 for a steel with a yield stress of 30,000 psi. for various ratios L/r and e/k_c. If γ be defined as the division factor related to the Engesser-Shanley stress, i.e.

$$\gamma = \sigma_{cE}/\sigma_{cr} \qquad (19)$$

it is seen directly that $\gamma = \gamma' (\sigma_{cE}/\sigma_{cK})$, where σ_{cE}/σ_{cK} follows from Fig. 6. The ratios so obtained are given in Fig. 12 by the solid curves. They can be approximated rather well by the equations

for $200 > L/r > 107$: $\gamma = 1 + \dfrac{12{,}200\,(e/k_c)^{2/3}}{(L/r)^2}$

$77 > L/r > 20$: $\gamma = (1 + .52\, e/k_c) + \dfrac{(e/k_c)^{1/2}}{100}\left(\dfrac{L}{r} - 20\right)$ (20)

$107 > L/r > 77$: interpolate between γ_{77} and γ_{107}

Expressions (20) hold only for the steel of Ref. 6 and are believed to be better approximations than given in Ref. 9. They are plotted in dashed lines in Fig. 12a.

No exact values for steels other than Chwalla's (σ_{ys} = 30,000 psi.) are available. Approximate data, assuming a steel with a stress-strain diagram that is straight up to the yield stress and horizontal from there on, are given in

Ref. 10 for steels with yield stresses of 34,000 and 51,000 psi. (2400 and 3600 kg/cm^2). Denoting the difference in γ found from Ref. 10 for these two steels by $\Delta\gamma$, the values for a steel with a yield stress of 50,000 psi. were calculated sufficiently accurately as

$$\gamma_{50,000} = \gamma_{38,000} + \frac{50,000 - 38,000}{51,000 - 34,000}\Delta\gamma$$

Fig. 12b shows the curves so obtained as solid lines. These are approximated fairly well by the formulas

for $200 > L/r > 90$: $\gamma = 1 + \dfrac{9500\,(e/k_c)^{4/3}}{(L/r)^2}$

$60 > L/r > 20$: $\gamma = (1 + 0.52\,e/k_c) + \dfrac{(e/k_c)^{1/2}}{120}(\dfrac{L}{r} - 20)$ (21)

$90 > L/r > 60$: interpolate linearly between γ_{60} and γ_{90}

which are given by the dashed curves in Fig. 11b. Interpolating linearly between Eq. (20) and (21) one obtains

for $200 > L/r > (L/r)_c$: $\gamma = 1 + (21,000 - 0.23\sigma_{ys})\dfrac{(e/k_c)^{2/3}}{(L/r)^2}$

$(L/r)_c - 30 > L/r > 20$: $\gamma = (1 + 0.52\,e/k_c) + \dfrac{(e/k_c)^{1/2}}{370 - 0.00\sigma_{ys}} \cdot (\dfrac{L}{r} - 20)$ (22)

$(L/r)_c > L/r > (L/r)_c - 30$: interpolate linearly between γ for $(L/r)_c - 30$ and $(L/r)_c$

where

$$(L/r)_c = 157 - 0.0014\,\sigma_{ys} \qquad (23)$$

Lastly, to generalize the plastic reduction factor η for steels of any yield point it is observed from Eq. (15) that for σ_{ys} = 38,000 psi. the slenderness ratio cL/r = 90 is that above which the column can be assumed to act elastically as far as the proposed Simplified Method is concerned. Since for a given buckling modulus the critical stress is inversely proportional to $(cL/r)^2$, the slenderness where plastic action begins to govern is about inversely proportional to the square root of the yield stress. Since for σ_{ys} = 38,000 psi. the limiting value is cL/r = 90, in general for steels with other yield stresses this limiting ratio will be

$$(cL/r)_b = 90\sqrt{\frac{38,000}{\sigma_{ys}}} \qquad (24)$$

so that in general from Eq. (15)

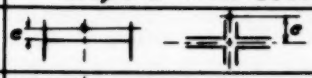
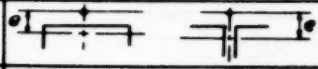
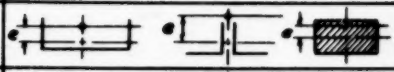
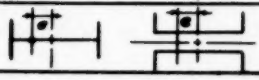
$$\left.\begin{array}{ll} \text{for} \quad cL/r > (cL/r)_b & : \eta = 1 \\ (cL/r)_b > cL/r > 20 & : \eta = (cL/r)\Big/(cL/r)_b \end{array}\right\} (25)$$

(c) Shape factor

In the preceding, only columns of rectangular cross section have been considered. An extension of the same methods to cross sections of other shapes would require graphs like Fig. 7 and formulas like Eq. (22) for several types of cross sections. Also the reduction factor η from Eq. (25) depends on shape.

A practical method for design, however, is to multiply the equivalent eccentricity ϵ with a shape factor which can be determined from Chwalla's calculations[11] and is given in Table 3. This table differs somewhat from the analogous table given on page 45 of Ref. 9.

Table. 3

	Shape of Cross Section	μ
1		0.8
2		1.25
3		1.0
4		1.4

(d) Summary of Simplified Method

The procedure for calculating the buckling stress for symmetrically loaded and symmetrically restrained columns of any cross-sectional shape is then as follows:

Let buckling member AB (Fig. 4a or b) have a length L, a moment of inertia I, a radius of gyration r and a core radius k_c, while I/L = K.

The procedure for a column of a steel with a given yield stress is now as follows:

1. Calculate the spring constant β of the elastic restraints given by the adjacent members to the buckling member AB, taking account of the axial forces in these members, if any, (Ref. 8).

2. Estimate the effective length of AB for concentric compression. For this length cL of the equivalent simply supported column (Fig. 5b) calculate η from Eq. (25), finding $(cL/r)_b$ from Eq. (24).

3. Calculate $(\beta/\eta EK)_{AB}$ and from it find the reduction coefficient c and thus cL from Graph II, Ref. 8.

4. Repeat the calculation of $\beta/\eta EK$ with the newly found slenderness ratio cL/r until this same effective slenderness results from Graph II, Ref. 8.

5. Calculate the equivalent eccentricity ε by distributing the external moments Pe without taking account of the axial forces in all members, i.e.

$$\varepsilon = \frac{2\eta EK}{2\eta EK + \beta_o} e$$

where β_o is the spring constant of the elastic restraints without taking account of axial forces.

6. Take account of the cross sectional shape by multiplying ε by the pertinent shape factor μ from Table 3 giving $\varepsilon_t = \mu\varepsilon$.

7. With values cL/r and ε_c/k_c find the eccentric buckling stress of the column as $\sigma_{cr} = \sigma_{cE}/\gamma$. Here σ_{cE} is given by Eq. (16) where L/r = cL/r and γ by Eq. (22), in which L/r = cL/r, $e/k_c = \varepsilon_c/k_c$ and $(L/r)_c$ is given by Eq. (23).

EXPERIMENTAL INVESTIGATION

(a) Specimens and testing procedure

The purpose of the experimental investigation was to determine for eccentrically-loaded, elastically-restrained steel columns the effect of eccentricity upon the ultimate load and upon the load at which outer fiber yield first occurred. Restraints were applied symmetrically. In order to assess the effect, if any, of residual rolling stresses, geometrically similar columns of both dead-annealed and as-rolled material were tested. Two types of cross-sections were used to fabricate the columns: one and one-half inch square bars and I-sections 4 I 9.5.

Compression stress-strain tests were performed on short lengths of the whole sections, in both annealed and unannealed condition. Typical results of these tests are presented in Fig. 13. The unannealed material is practically identical to the annealed, except that the yield point is higher. A minor effect of residual stress is indicated by the rounded knee of the curve for the unannealed I-sections.

The column specimens had slenderness ratios (L/r) of about 80 based on the total free length between knife-edges. The computed effective slenderness ratio (cL/r) for concentric buckling reported in Table 4, is smaller than this free length due to the elastic restraints imposed at the ends of the column. Four sets of columns were tested with the ratio of eccentricity to radius of gyration (e/r) varying from 0 to 2 for each set, as follows:

1-1/2 inch square bars, annealed
1-1/2 inch square bars, unannealed
4 I 9.5 I-sections, annealed
4 I 9.5 I sections, unannealed

Geometric properties are given in Table 5.

The general test setup is shown in Fig. 14. The elastic end-restraining fixtures are seen to consist of a pair of channels forming the stationary part (connected to the head or table of the testing machine) and a standard rolled wide-flange section forming the movable part, the two parts being separated by a knife-edge support at their centers and held together elastically at their ends by precompressed springs. An eccentric load applied to the fixture causes a rotation about the knife-edge of the wide-flange section relative to the double-channel beam, against the resistance afforded by the springs.

Lateral deflections were measured by dial gages at the center and quarter-points of the columns, and at the upper

head of the testing machine. Rotations at the stations indicated in Figure 14 were measured by means of a portable inclinometer.

It is thought that the centering procedure for columns supported as in these tests is sufficiently unusual to warrant description in some detail. The elastic center-deflection and end-rotations of a given column have unique values for a given load and eccentricity and may be computed by means of customary elastic analysis. (See Ref. 14) They must coincide with measured values of these quantities if the centering of the column is to conform to assumed conditions of support and eccentricity. Each column was located in the testing machine to a predetermined eccentricity, measured from accurately located knife-edge positions, and the location so obtained was checked and corrected by centering under load. To assure correct eccentricities at both ends, center deflections and end-rotations were measured at selected loads in the elastic range, and centering was adjusted until measurements coincided with precomputed values. The eccentricity at each end must be checked individually and requires that the centering be done by both deflection and rotation; since a variety of paired values of end eccentricities can result in the same center deflection, it is impossible to center by deflection

only. Furthermore, the proper combination of center deflection and end rotations cannot be obtained unless the measured end eccentricity is correct, providing a check on and means of adjustment of the predetermined location of the column.

The centering procedure was as follows:

(1) Center the column in the direction along the knife-edges at a load not more than one-third of the load at which outer fiber yield is expected to occur. This was controlled by strain measurements read with four 1-inch Tuckerman optical strain gages located on the extreme fibers at the quarter-points of the column. Equalizing the strains was accomplished by shimming, of the order of 0.005 inches, usually between the column base-plate and the WF section and more infrequently between the elastic end fixture and the machine head surface when the required shim thickness exceeded 0.005 inches. Centering in this direction was considered sufficiently accurate when the average strains on the two opposing column faces did not differ from each other by more than 5% of the average strain on the whole tranverse section.

(2) Center the column in the direction perpendicular to the knife-edge, that is, in the direction of the eccentricity at a load not more than one-third of the expected

yield load. This was controlled in several ways: (a) the central deflection to coincide with its precomputed value, (b) the quarter-point deflections to be equal, (c) the end rotations to be equal and opposite and coincident with their precomputed value, (d) the strains at the upper quarter-point to be in reasonable balance with those at the lower quarter-point. The latter was intended only as a check, and was considered less important than meeting the first three conditions. Adjustments were made by tightening the springs of the elastic end fixtures to change the initial inclination of the WF section in the direction desired, and, if necessary, by changing slightly the distance between the knife-edge and column axis.

(3) Check the centering of the column in both directions as in Steps (1) and (2) at loads up to about two-thirds of the load at which outer fiber yield is expected to occur, making such adjustments as necessary. First centering must be done at relatively low loads to avoid accidental overstressing, whereas final centering should be carried as close to the end of the elastic range as is considered safe.

The centered column was then tested by measuring deflections and rotations at close intervals at loads up to and well beyond the ultimate load.

Ordinary whitewash was coated on each column before

testing to serve as a visual indicator of yielding. Flaking of the whitewash allowed easy and direct observation of initial as well as progressive yielding. The whitewash technique was suggested to the Cornell investigators by their Lehigh University colleagues, particularly Dr. L.S. Beedle.

(b) Results and discussion

Experimental and theoretical values of ultimate mean stress and the mean stress at which outer fiber yield occurred are presented and compared in Table 4 and in Figs. 15 and 16. The mean stress is in all cases the axial load on the column divided by the nominal total area of the column. It will be noticed that, as a direct consequence of higher yield point, the column strength of the unannealed specimens is uniformly higher than that of the annealed ones.

Theoretical values of ultimate mean stress have been computed by the Simplified Method given earlier in this paper.

For the I-sections, the theoretical values have been computed for a shape factor of 1.0 and also for a shape factor of 0.8 to account for the influence of the shape of the cross-section. Values for both shape factors are given on Fig. 15; the data in Table 4 refer to a shape factor of 0.8. For the square sections the experimental

ultimate loads (ultimate mean stresses) are seen to be practically identical with the theoretical values. For the I-sections, the experimental values are very close to the theoretical values computed with a shape factor of 0.8, with slight scatter on the conservative (high) side. All experimental values are significantly higher than the theoretical values computed for a shape factor of 1.0

Mean stress at incipient outer fiber yield is presented in Table 4 and Figs. 15 and 16. Experimental values were obtained from (a) observation of flaking of the whitewash coating on the column, and (b) by statical computation, denoted as "semi-empirical", based on measured end-rotations and center-deflections from the tests. With the measured center deflection and the simultaneous measured end rotations (and therefore the corresponding end-restraining moments), the center moment may be computed statically for a given load. From this bending moment and the simultaneous axial load, the maximum fiber stress may be computed according to $\sigma = \frac{P}{A} + \frac{Mc}{I}$. The value of P/A corresponding to a computed fiber stress equal to the yield point is the "semi-empirical" mean stress at incipient yield.

Both the experimental and the semi-empirical values have been compared with theoretical values computed by a

successive approximation procedure based on customary elastic analysis. (See Refs. 14 and 2)

Both the semi-empirical and the theoretical values have been based upon the lower yield point (as observed from the mechanical properties tests) and agree quite closely with each other. However, the whitewash indication was, in general, slightly higher than either the theoretical or semi-empirical value. The flaking of the whitewash is probably delayed beyond the point of incipient yielding until the formation of yield lines has progressed some distance inward from the edge of the flange, and the actual formation of yield lines may be inhibited by the existence of an upper yield point for the material. Both effects will result in a whitewash indication of incipient yielding at a load somewhat higher than that predicted theoretically on the basis of lower yield point. That the upper yield point can exist as a definite characteristic related to the shape of the cross-section, even at slow rates of loading, has been shown theoretically in Refs. 12 and 13. Due to this latter cause alone, formation of yield lines can be inhibited until a stress some 15-20 percent above the lower yield point has been reached, and this seems to be the most plausible way of accounting for the major portion of the definite deviation shown in Figures 15 and 16.

Theoretical curves have been drawn for such inhibited yield based on an extreme fiber stress 15% above the lower yield point of the material. The inhibiting of yielding occurs for most of the specimens, and is most markedly regular for the unannealed I-sections. The whitewash indication is somewhat erratic for the annealed specimens, presumably as a result of loosening and irregular shedding of scale during the annealing process.

Typical load-deflection and load-rotation curves are given in Fig. 17a. For small eccentricities, up to $e/r = 1$, deflections at ultimate (maximum) load are relatively small, generally less than about 1/100 th of the column length. For larger eccentricities, however, deflections at ultimate loads are significantly larger, of the order of 1/50 th of the length.

There may be some doubt whether design determinations legitimately should be based on loads accompanied by such large deflections. At loads corresponding to outer fiber yield, as shown in Fig. 17a, deflections range between 1/225 th and 1/450 th of the length respectively for the largest and smallest non-zero eccentricities. If criteria for limit loads were to be correlated with deflections, the latter may seem permissible while the former may not. In other words, deflections at ultimate loads may be so large

as to impair the usefulness of a structure, whereas deflections at loads which produce incipient yielding are of little practical consequence.

The typical load - end rotation curves of Fig. 17b show that with increasing load the end rotations increase at a faster rate than the loads, the more so the closer the ultimate load is approached. The share of the total applied moment Pxe which is resisted by the end restraints is equal to "rotation times spring constant". Hence, this share of the applied moment increases with increasing load so that a progressively smaller portion of the applied moment Pxe is resisted by the column. This behavior is easily demonstrated analytically in the elastic range (see pp. 10, 11 of Ref. 8). It is accentuated in the plastic range as can be seen from Eq. (8). In fact, as the load increases the plastic reduction factor decreases gradually from unity to its final value η so that the effective eccentricity ε becomes a progressively smaller portion of the applied eccentricity e.

From this reasoning it is important to recognize, practically, that a given amount of end eccentricity "e" is of much less detrimental consequence for an end-restrained column than it is for a hinged column. This is a further reason why design provisions based on the performance of simply supported, eccentrically loaded columns tend to be over-conservative.

CONCLUSIONS

(1) Ultimate or buckling loads of eccentrically loaded, elastically restrained columns can be computed rigorously only on the basis of plastic action, by an extremely lengthy grapho-analytical method which does not lend itself to use in practical design work.

(2) An approximate, simplified method is given in the paper which reduces the amount of calculating work to reasonable limits.

(3) The results of this method are in good agreement with the presented experimental evidence. It should be pointed out, though, that these tests cover only a limited range of variables (L/r, degree of restraint, magnitude of eccentricity).

(4) The ultimate loads of such columns are greatly in excess of those loads which cause incipient outer fiber yielding, up to almost 100% in the present tests, depending on eccentricity. In this connection attention is drawn to the fact that the test sections were of very compact shape and not subject to local buckling. It is possible that for sections with relatively thin flanges the difference between ultimate and incipient yield loads will be smaller than for compact sections (see Refs. 15, 16).

(5) If design provisions are to be based on ultimate loads rather than on those which cause incipient yielding, as has been advocated (see e.g. Ref. 9), even the presented simplified method may be too cumbersome for practical use. Column design would have to be done by the use of charts and/or tables, as is customary in other fields, such as reinforced concrete. The methods presented in this paper may serve to compute such charts or tables.

(6) In considering the merits of establishing ultimate loads as the basis for design criteria it should be noted that when ultimate loads are reached column deflections are excessively large, particularly for the larger eccentricities. In contrast, these deflections are of moderate magnitude at those loads which cause incipient yielding.

(7) In elastically restrained columns a portion of the applied end moment Pxe is resisted by the end restraints so that the column resists only the remaining portion. The fraction of the total end moment carried by the column decreases with increasing load, both in the elastic and in the plastic range. It follows that a given amount of end eccentricity is much less detrimental to a restrained column than it would be for the same column without restraints. Design provisions based on simply supported, eccentrically

loaded columns, therefore, tend to be over-conservative if applied to compression members which are parts of rigid frameworks.

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Table 4 - Summary of Eccentric Column Tests

Spec. No.	e/r	cL/r	Ultimate Mean Stress, ksi			Mean Stress at outer fiber yield, ksi				
			Experimental	Theoretical	Ratio (4)/(5)	Whitewash	Semi-empirical	Theoretical	Ratio (7)/(9)	Ratio (8)/(9)
(1)	(2)	(3)	(4)	(5) **	(6)	(7)	(8)	(9)	(10)	(11)
7AI	0	50.2	29.00	27.11	1.070	27.20	30.0	30.0	0.907	1.000
6AI	0.5	50.2	24.05	22.65	1.061	23.75	17.88	17.75	1.338	1.006
2AI	1.0	50.2	20.65	20.05	1.030	13.22	13.48	13.29	0.995	1.015
5AI	1.5	50.2	18.40	18.05	1.019	11.98	10.62	10.75	1.114	0.988
3AI	2.0	50.2	17.80*	16.45	1.081	10.15	9.21	9.05	1.121	1.018
12UI	0	51.8	34.65	34.25	1.012	30.80	38.0	38.0	0.812	1.000
15UI	0.5	51.8	29.35	27.80	1.056	23.95	21.93	21.77	1.110	1.008
13UI	1.0	51.8	26.0	24.25	1.071	19.90	16.44	16.30	1.220	1.010
16UI	1.5	51.8	21.45	21.70	0.990	15.59	13.10	13.21	1.180	0.992
14UI	2.0	51.8	19.60	19.45	1.007	14.50	11.39	11.13	1.302	1.022
8AS	0	39.6	32.20	29.92	1.076	32.20	32.20	32.20	1.000	1.000
11AS	0.5	39.6	27.00	27.00	1.000	25.55	24.30	23.00	1.110	1.057
10AS	1.0	39.6	24.20	24.93	0.972	21.10	19.77	18.52	1.140	1.066
9AS	2.0	39.6	22.60	21.80	1.037	15.55	14.12	13.53	1.149	1.042
17US	0	40.3	37.78	36.24	1.040	31.10	39.00	39.00	0.797	1.000
20US	0.5	40.3	32.60	32.17	1.013	28.65	27.22	27.40	1.046	0.995
21US	1.0	40.3	30.20	29.55	1.022	24.40	22.55	22.03	1.107	1.022
19US	2.0	40.3	26.00	25.69	1.011	17.10	16.30	16.10	1.062	1.012

CODE: A, annealed; U, unannealed; I, I-section; S, square bar

* Ultimate load probably too high - plastic equilibrium not reached

** Item (5) computed with shape factor 0.8 for I-sections, 1.0 for square bars

Table 5

Geometric Properties of Column Specimens

	1 1/2 in. sq. bars	I-Sections 4I9.5
Clear or Flexible length	27.00"	42.00"
Distance out-to-out of end blocks	29.95" ± .05"	44.95" ± .05"
Distance between knife-edges	34.95" ± .05"	49.95" ± .05"
Nominal cross-sectional area	2.25 sq. in.	2.76 sq. in.
Nominal radius of gyration	0.433"	0.580"

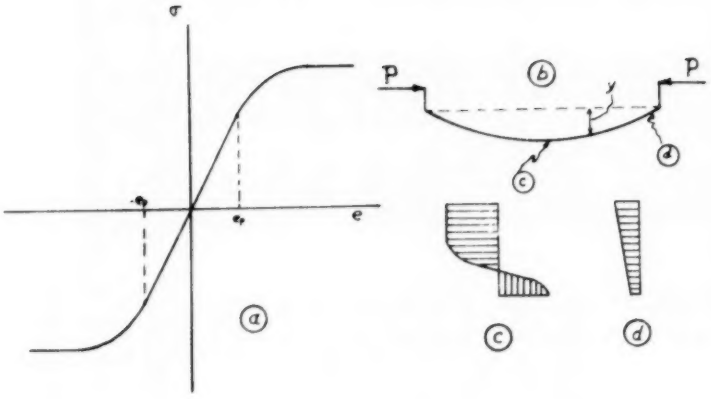

Fig. 1

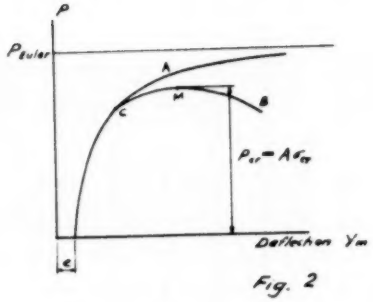

Fig. 2

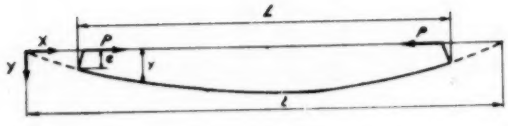

Fig 3

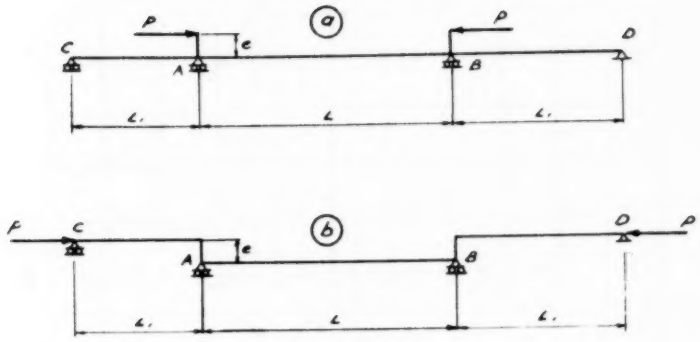
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Fig. 4

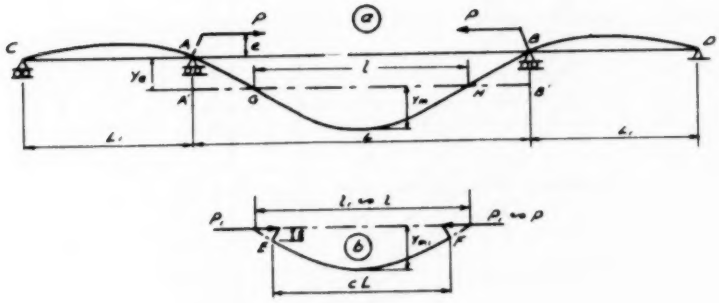
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Fig. 5

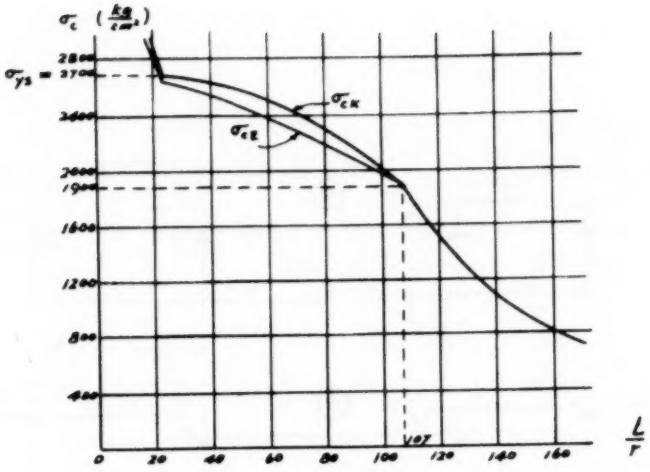


Fig. 6

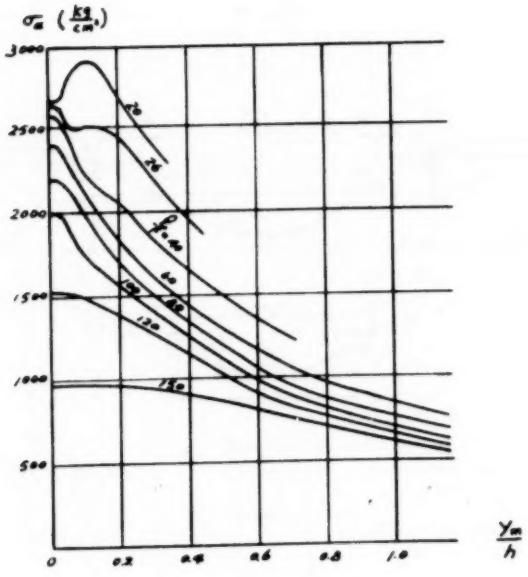


Fig. 7

292 bis

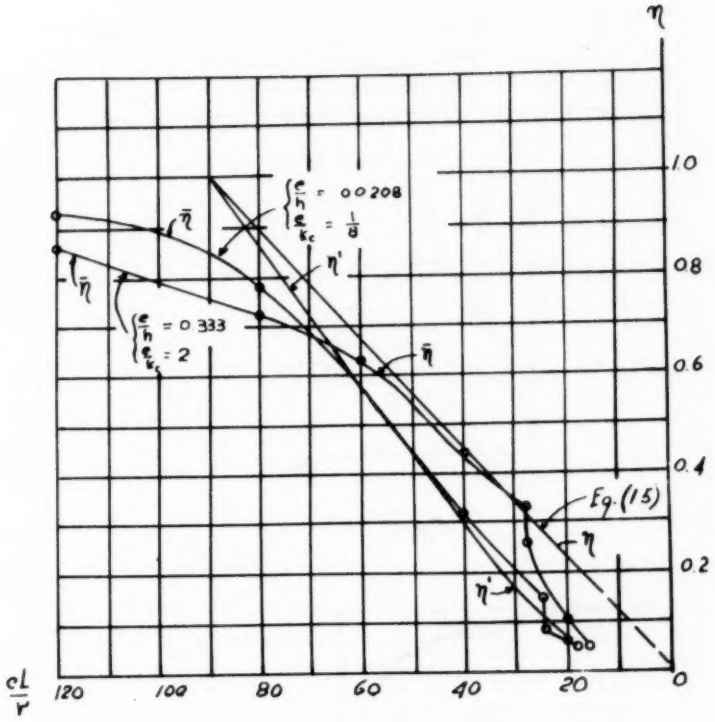

η
1.0
0.8
0.6
0.4
0.2
0
120
100
80
60
40
20
$\frac{cL}{r}$
$\left\{\frac{e}{h} = 0.0208\right.$
$\left.\frac{e}{k_c} = \frac{1}{\theta}\right.$
$\bar{\eta}$
η'
$\left\{\frac{e}{h} = 0.333\right.$
$\left.\frac{e}{k_c} = 2\right.$
Eq. (15)

Fig. 8

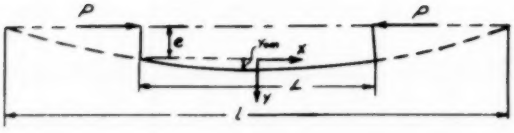


Fig. 9

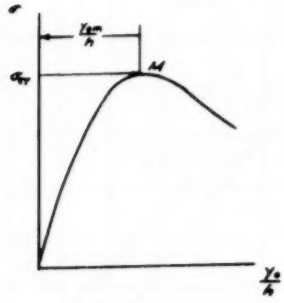


Fig. 10

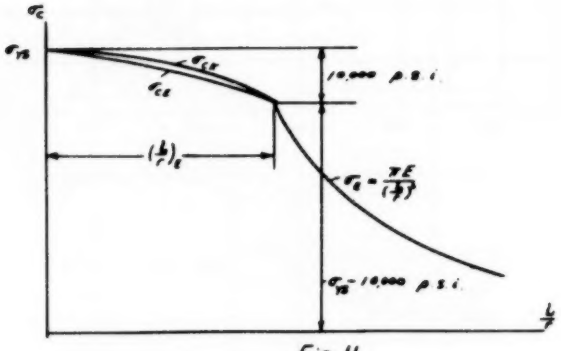


Fig. 11

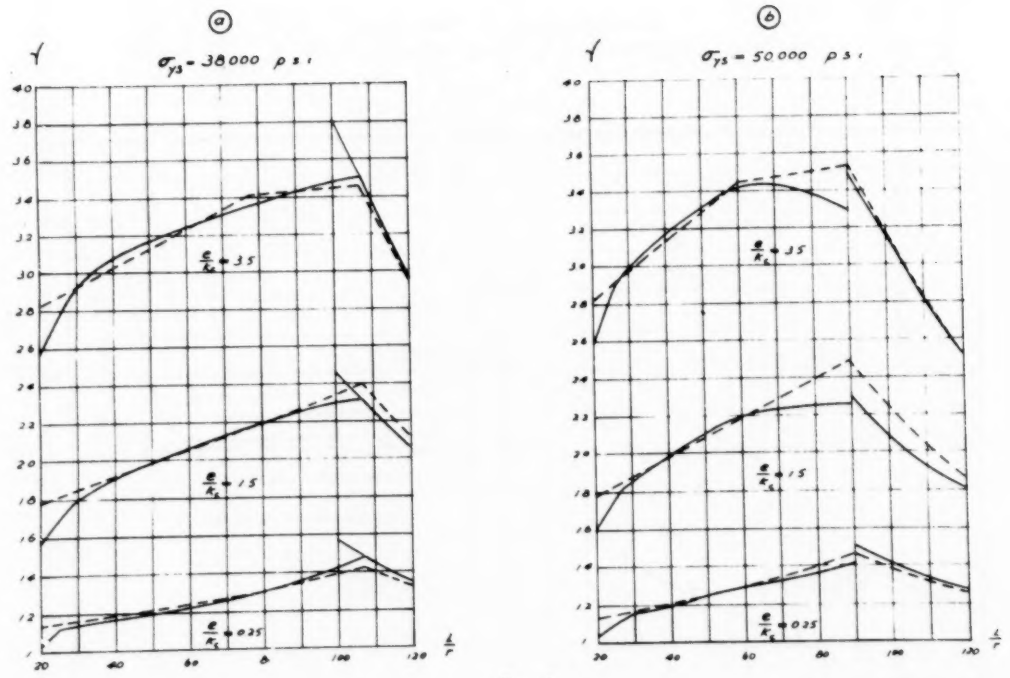

(a)
σ_{ys} = 38,000 p.s.i.
$\frac{e}{k_s}$ = 35
$\frac{e}{k_s}$ = 15
$\frac{e}{k_s}$ = 0.25
$\frac{L}{r}$
(b)
σ_{ys} = 50,000 p.s.i.
$\frac{e}{k_s}$ = 35
$\frac{e}{k_s}$ = 15
$\frac{e}{k_s}$ = 0.25
$\frac{L}{r}$

Fig. 12

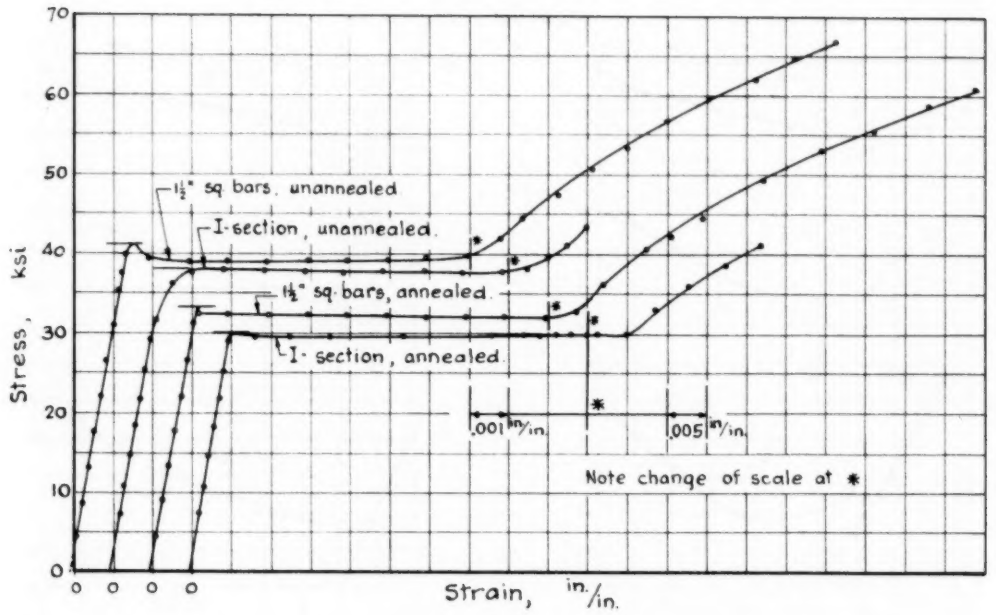
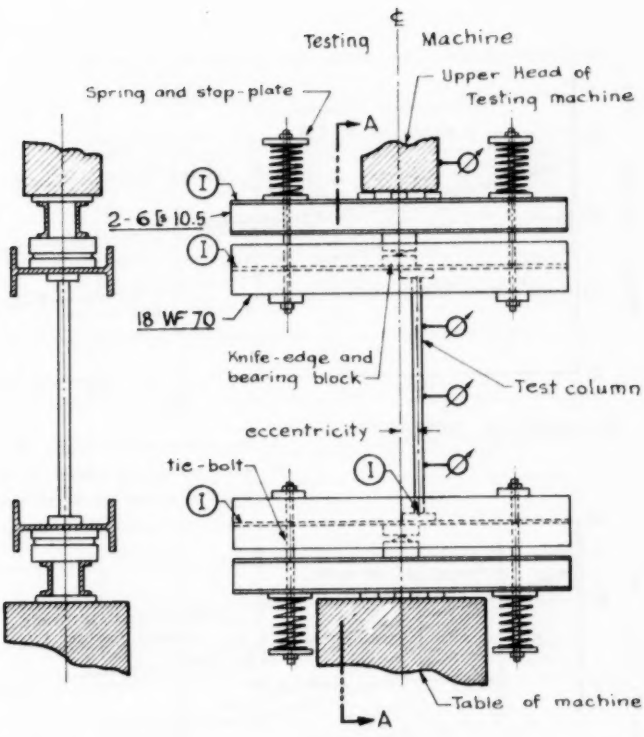


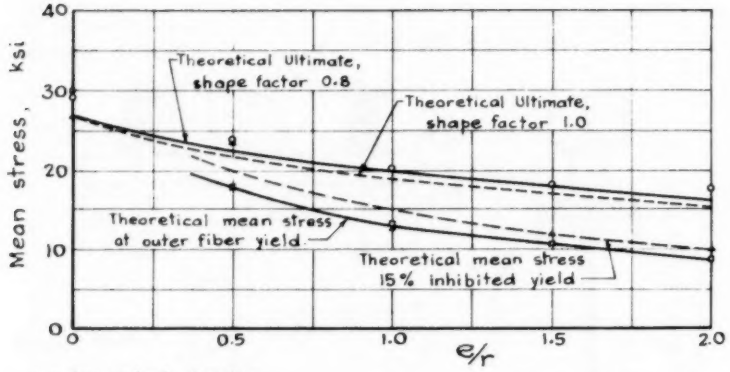
Fig. 13 - Typical Stress-Strain Curves For Whole Sections in Compression.



- deflection dial gage

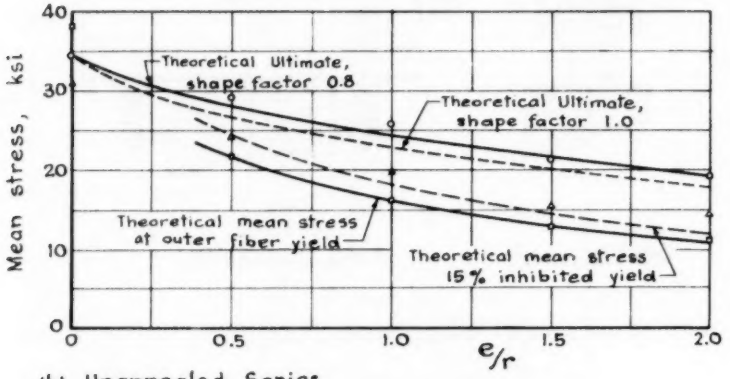
(I) - inclinometer station

Fig. 14 - SCHEMATIC DIAGRAM OF TEST ARRANGEMENT.



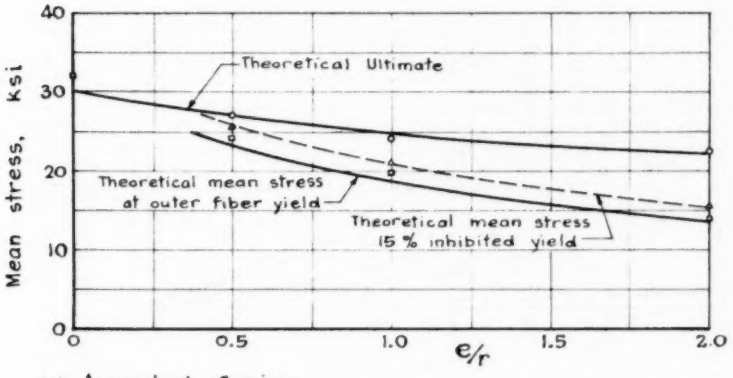
(a) Annealed Series.

- o Experimental Ultimate Mean Stress.
- □ Semi-empirical mean stress at yield.
- △ Mean stress at yield, whitewash indication.



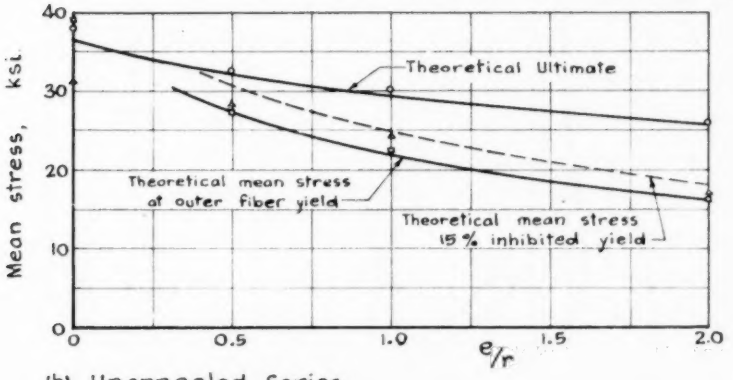
(b) Unannealed Series.

Fig 15 - Test Results, I-sections 4 I 9.5



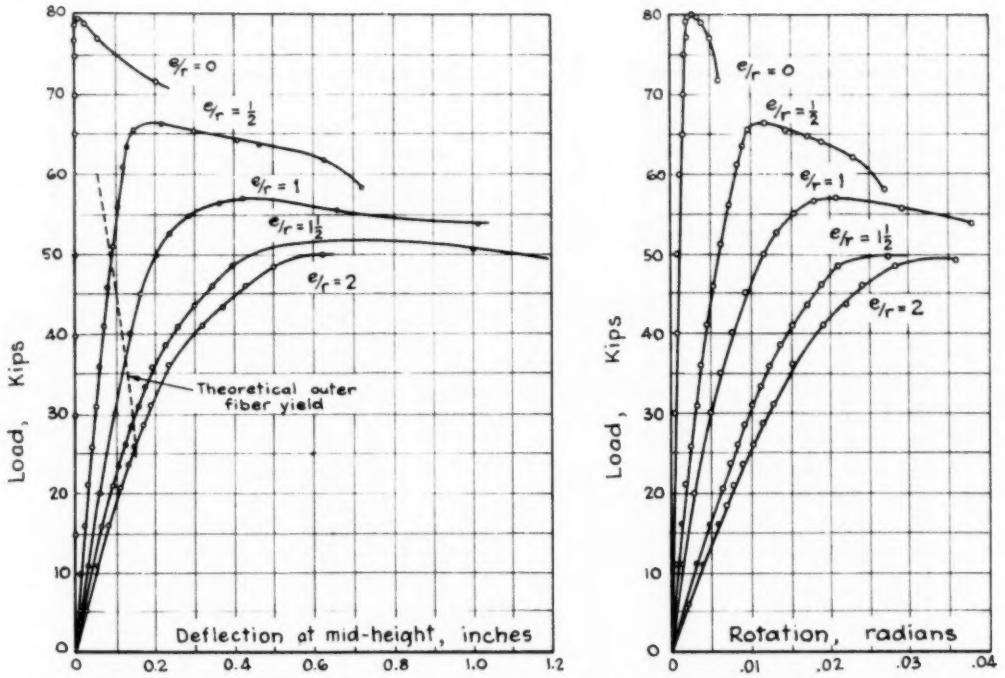
(a) Annealed Series.

o Experimental Ultimate Mean Stress.
□ Semi-empirical mean stress at yield.
△ Mean stress at yield, whitewash indication.



(b) Unannealed Series

Fig. 16 - Test Results, 1½-inch Square Bars



(a) Center-deflection (b) End-rotation

Fig. 17 - Column Deformation Curves For I-sections 4I9.5, Annealed.

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